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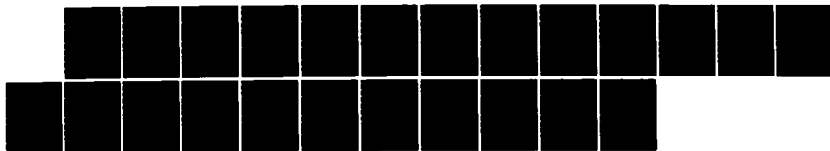
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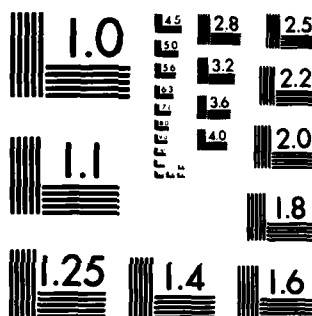
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Automatic Symbolic Solution of Markov Chains

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Automatic Symbolic Solution of Markov Chains*

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Abstract

Continuous time Markov chains are commonly used in system performance modeling. Increasing system complexity and non-Markovian behavior can drastically increase the size of a Markov model's state space. Accordingly, approximation techniques have been introduced to reduce the resources needed to solve Markov chain models. In this paper we discuss a method for automatically deriving symbolic solutions of Markov chains. Symbolic solutions should provide insight when attempting to evaluate the validity of both Markov models and approximation techniques for their solution.

1. Introduction

Continuous time Markov chains (CTMC) are commonly used tools in computer systems modeling. CTMC have been used to model program behavior,¹ system performance,^{2,3} system reliability,^{4,5} and system availability,⁶ and also in the combined evaluation of performance and reliability.^{7,8} Although the limitation of exponentially distributed state occupancy times, as implied by a homogeneous CTMC, appears to be restrictive, it is possible to use the Coxian method of stages to allow arbitrary phase type distributions.^{9,10,11}

In general, once a Markov chain model of a system has been constructed, there are several solution methods available. Figure 1 summarizes these methods and typical modeling packages that employ them. The Markov model of a system can be

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solved using integral equations, formally taking the convolution of the probability of entering the state with the probability of remaining in it.^{12,13} Alternatively, the Markov chain can be converted to a coupled set of homogeneous differential equations.³ This set of equations can be solved using either numerical techniques or Laplace transforms. A third solution method is simulation. As an introduction, we will briefly discuss some of the advantages and disadvantages of each of these methods.

Simulation can be more realistic than analytic models. Many types of complex systems, particularly those for which independence assumptions are invalid, can be directly modeled using simulation. However simulation models can have high development costs and may require large amounts of computer time to obtain statistically significant results. Therefore, the low cost alternative of analytic modeling can be attractive, even if approximating assumptions are necessary. If a range of systems must be compared, similar systems must often be simulated individually. Analytic modeling may permit the comparison of such systems without repeated simulation.

The most common analytic approach is to represent the Markov chain as a set of coupled differential equations. Each equation describes the flow "balance" conditions in a corresponding state of the chain, i.e. the instantaneous rate of change in the probability of being in a state is equal to the rate of arrival into the state less the rate of departure from the state at that instant. This set of equations can be solved using either Laplace transforms or numerical techniques. Using Laplace transforms for large systems (either numerically or symbolically) may require finding the roots of many large polynomials, a computationally expensive task. One advantage of numerical techniques is that they can be easily extended to evaluate non-homogeneous Markov chains. However, if the system is stiff, i.e. if two or more transition rates out of any single state differ greatly in magnitude, special care will be needed to get an accurate solution.

Sets of integral equations are similar to coupled differential equations.¹² Integral equations provide a basis for modeling both semi-Markov processes and non-homogeneous Markov chains. Integral equations for Markov chain state probabilities will also provide the basis for the closed form solution techniques discussed later in this paper.

One issue we do not address in detail is the solution of cyclic Markov chains. In the context of modeling fault-tolerant systems, we are restricting our attention to non-repairable systems. Such systems can generally be represented by Markov chains without cycles. If cycles are present in the model, all the solution methods discussed are of diminished utility. The simulation of a cyclic Markov chain may be more expensive (for the same degree of accuracy) than for an acyclic chain of the same size, as the number of possible paths through the chain are no longer finite. If an analytic solution of a cyclic Markov chain is desired, numerical solution of systems of either differential or integral equations is usually the recommended approach. Approximation techniques may still allow us to obtain a symbolic solution, albeit an inexact one.

To illustrate the use of Markov models, we consider an example from reliability modeling.¹⁴ Figure 2 depicts the transition diagram of a Markov chain representing a 3-component parallel redundant system. The individual components have lifetimes that are independent and exponentially distributed with parameter λ . When an individual component fails, a reconfiguration process with rate parameter δ begins. This process is guaranteed to reconfigure the system as long as a second fault does not occur before the reconfiguration is completed. The reliability of the system at time t , denoted $R(t)$, is given by $1 - P\{\text{process in state } F_2 \text{ or } F_3 \text{ at time } t\}$.

A package employing traditional (i.e. numerical) solution techniques would input numeric values for the parameters of the model (here δ and λ) and would solve the system numerically for t less than some fixed value. To determine the actual behavior

of the system as a function of a parameter other than t would require many runs of such a program.

In this paper, we discuss a method for the derivation of state probabilities of an acyclic Markov chain in a symbolic fashion. Closed form results (that previously could be obtained only by hand) give greater insight into actual system behavior by allowing us to easily study the relationship between input parameters and the resulting state probability distributions. Our approach is based on the use of integral equations. It is computationally similar to using Laplace transforms to solve systems of coupled ODE's. The program implementing the algorithm discussed in the paper is called ACE (*Acyclic Markov Chain Evaluator*). The solution of cyclic chains (which presents additional difficulties) is omitted from our discussion.

In section 2, we describe our method which is partially inspired by the program SPADE.¹⁶ In section 3 we describe our program's implementation. Some examples of the use of ACE are given in section 4.

2. Basic Approach

Consider an acyclic continuous time Markov Chain. Let the states be numbered $1, 2, \dots, N$. Let $\lambda_1, \lambda_2, \dots, \lambda_M$ be the transition rate variables. The transition rate from state i to state j is denoted by q_{ij} , where q_{ij} can be expressed as a linear sum of transition rate variables, i.e.

$$q_{ij} = \sum_{k=1}^M c_{(ij)k} \lambda_k \quad c_{(ij)k} \in (-\infty, \infty). \quad (1)$$

Further let $q_i = \sum_j q_{ij}$ denote the total exit rate from state i .

For any state i of an acyclic Markov chain, let $P_i(t)$ be the probability that the system is in state i at time t . For any state i , $P_i(t)$ may be written as a polynomial of the form

$$P_i(x) = \sum_j e^{\gamma_j x} \left[\sum_k a_{jk} x^k \right]. \quad (2)$$

The fact that the state probability distributions are of this form is easily derived. First observe that the initial state(s) probability(ies) has (have) this form. The probability of being in (non-initial) state i at time t can be written as:

$$P_i(t) = \int_0^t \sum_{j \in J(i)} P_j(x) q_{ji} e^{-q_i(t-x)} dx \quad (3)$$

where $J(i)$ is the set of states with a transition leading to state i . By induction, it is easy to show that, if every $P_j(x)$ has the form (2), then so does any $P_i(x)$ derived using (3).

We now derive the equations needed to calculate the constants of equation (2) for any state of an actual acyclic Markov chain. Let $S(i)$ be the set of poles of the Laplace-Stieltjes Transform of $P_i(t)$, i.e. the set of the γ_j 's of (2). Let us rename $\gamma^* \triangleq -q_i$ and define $S(J(i)) \triangleq \bigcup_{j \in J(i)} S(j)$. Setting $N(j) = |S(j)|$, we can write

$S(j) = \{\gamma_{j1}, \gamma_{j2}, \dots, \gamma_{jN(j)}\}$. If we number the poles then $P_j(t)$ may be written as:

$$P_j(t) = \sum_{l=1}^{N(j)} e^{\gamma_{jl} t} \left[\sum_{k=0}^{L(j,l)} a_{jlk} t^k \right] \quad (4)$$

$L(j,l)$ is the maximum power of t associated with pole γ_{jl} in $P_j(t)$ such that $a_{jlk} \neq 0$.

If $N(j) > 1$, it is easy to show using an inductive proof that, for any pole γ_{jl} ,

$$(a_{jll(j,l)} \neq 0) \Rightarrow \forall k < L(j,l) (a_{jlk} \neq 0) \quad (5)$$

If $N(j)=1$, then $P_j(t)$ is of the form $at^k e^{\gamma t}$. This corresponds to the case where there is only one directed path from the original state to state j and all the transition rates along this path are all equal, i.e. $q_j = -\gamma$. Only in this case is the implication in (5) not satisfied. Thus $P_i(t)$ may be written

$$P_i(t) = \sum_{j \in J(i)} \sum_{l=1}^{N(j)} \int_0^t e^{\gamma_{jl} x} \left[\sum_{k=0}^{L(j,l)} a_{jlk} x^k q_{ji} \right] e^{\gamma^*(t-x)} dx. \quad (6)$$

But, especially when the $P_j(t)$'s have common poles, i.e. when $|S(J(i))| < \sum_{j \in J(i)} |S(j)|$.

this expression reduces to

$$P_i(t) = \sum_{\gamma \in S(J(i))} \int_0^t e^{\gamma x} \left[\sum_{j \in J(i)} \sum_{l=1}^{N(j)} 1_{(\gamma_{j,l} = \gamma)} \left(\sum_{k=0}^{L(j,l)} q_{ji} a_{jlk} x^k \right) \right] e^{\gamma^*(t-x)} dx$$

$$= \sum_{\gamma \in S(J(i))} \int_0^t \sum_{k=0}^{L_i(\gamma)} a_{\gamma,k} x^k e^{\gamma x} e^{\gamma^*(t-x)} dx \quad (7)$$

where

$$a_{\gamma,k} \triangleq \sum_{j \in J(i)} \sum_{l=1}^{N(j)} 1_{(\gamma_{j,l} = \gamma)} q_{ji} a_{jlk}, \quad \gamma \in S(J(i)) \quad (8)$$

and

$$L_i(\gamma) = \max_{j \in J(i)} \{ L_{j,l} \mid \gamma_{j,l} = \gamma \}. \quad (9)$$

Moving the integral inside the summation we obtain

$$P_i(t) = \sum_{\gamma \in S(J(i))} \sum_{k=0}^{L_i(\gamma)} e^{\gamma^* t} a_{\gamma,k} \int_0^t x^k e^{(\gamma - \gamma^*)x} dx. \quad (10)$$

The resolution of this integral depends on whether γ differs from γ^* . If $\gamma = \gamma^*$, then,

$$\int_0^t x^k e^{(\gamma - \gamma^*)x} dx = \int_0^t x^k dx = \frac{t^{k+1}}{(k+1)}. \quad (11)$$

Otherwise, when $S = (\gamma - \gamma^*)$, integration by parts yields

$$\int_0^t x^k e^{Sx} dx = e^{St} \left[\sum_{l=0}^k (-1)^l \frac{k!}{(k-l)!} \frac{t^{k-l}}{S^{l+1}} \right] + (-1)^{k+1} \frac{k!}{S^{k+1}}. \quad (12)$$

We can now write the equation for $P_i(t)$

$$P_i(t) = \sum_{\gamma \in S(J(i))} \left\{ 1_{(\gamma \neq \gamma^*)} \sum_{k=0}^{L_i(\gamma)} \left[e^{\gamma^* t} a_{\gamma,k} \left(\sum_{l=0}^k (-1)^l \frac{k!}{(k-l)!} \frac{t^{k-l}}{S^{l+1}} \right) \right. \right.$$

$$\left. \left. + a_{\gamma,k} (-1)^{k+1} \frac{k!}{S^{k+1}} e^{\gamma^* t} \right] + 1_{(\gamma = \gamma^*)} \sum_{k=0}^{L_i(\gamma^*)} e^{\gamma^* t} a_{\gamma^*,k} \frac{t^{k+1}}{(k+1)} \right\}. \quad (13)$$

We note that

$$S(i) = \{\gamma^*\} \cup S(J(i)) \quad (14)$$

and define

$$L(i, l) = \begin{cases} L_i(\gamma_l) & \text{if } \gamma_l \neq \gamma^* \\ L_i(\gamma_l) + 1 & \text{if } \gamma_l = \gamma^* \text{ and } \gamma^* \in S(J(i)) \\ 0 & \text{if } \gamma_l = \gamma^* \text{ and } \gamma^* \notin S(J(i)) \end{cases} \quad (15)$$

From (13) note that, if $u = k - l$, we have

$$\begin{aligned} \sum_{k=0}^{L_i(\gamma)} a_{\gamma, k} \left[\sum_{l=0}^k (-1)^l \frac{k!}{(k-l)!} \times \frac{t^{k-l}}{S^{l+1}} \right] \\ = \sum_{k=0}^{L_i(\gamma)} a_{\gamma, k} \left[\sum_{u=0}^k (-1)^{k-u} \frac{k!}{u!} \times \frac{t^u}{S^{k+1-u}} \right] \\ = \sum_{u=0}^{L_i(\gamma)} t^u \sum_{k=u}^{L_i(\gamma)} a_{\gamma, k} (-1)^{k-u} \frac{k!}{u!} \times \frac{1}{S^{k+1-u}} \end{aligned}$$

If we write $P_i(t)$ as

$$P_i(t) = \sum_{l=1}^{N(i)} e^{\gamma_l t} \sum_{k=0}^{L_{u,l}} a'_{\gamma_l, k} t^k \quad (16)$$

then

$$a'_{\gamma, u} = \sum_{r=u}^{L_i(\gamma)} (-1)^{r-u} \frac{r!}{u!} \times \frac{a_{\gamma, r}}{S^{r+1-u}}.$$

If $\gamma_l \neq \gamma^*$, we have

$$a'_{\gamma_l, k} = \sum_{r=k}^{L_i(\gamma_l)} (-1)^{r-k} a_{\gamma_l, r} \cdot \frac{r!}{k!} \frac{1}{(\gamma_l - \gamma^*)^{r-k+1}} \quad k=0, \dots, L_i(\gamma_l) \quad (17)$$

If $\gamma^* \notin S(J(i))$ then

$$a'_{\gamma^*, 0} = \sum_{l=1}^{N(i)} 1_{(\gamma_l \neq \gamma^*)} \sum_{k=0}^{L_i(\gamma_l)} (-1)^{k+1} \cdot k! \frac{a_{\gamma_l, k}}{(\gamma_l - \gamma^*)^{k+1}} \quad (18)$$

Otherwise, if $\gamma^* \in S(J(i))$,

$$a'_{\gamma^*, k+1} = \frac{a_{\gamma^*, k}}{(k+1)} \quad k=0, \dots, L_i(\gamma^*) \quad (19)$$

With these equations, we can easily compute the coefficients of the polynomials in t that multiply the exponentials in the state probability expressions.

3. Implementation

In this section we outline the procedure used by the ACE program to compute state probability expressions for an acyclic Markov Chain. Techniques for chains with cycles are more complex, requiring either the symbolic solution of a set of equations or some form of approximation. We also briefly discuss the operation and user interface of ACE.

The ACE procedure is detailed in the Appendix; we briefly outline it here. First the states are sorted according to the partial order induced by the transitions. For state i , the probability expression is computed by first determining all the poles of states that have transitions leading to state i . If the pole associated with state i 's outgoing transition is not in the incoming set, coefficients for the polynomials multiplying the incoming exponential terms are computed using equation (17). The new pole's polynomial multiplier is a constant computed using equation (18). If the outgoing pole is also in the set of incoming poles, the degree of its polynomial will be incremented. The new coefficients for the incremented polynomial can be computed using equation (19).

ACE is being developed as the first stage of a testbed for aggregation techniques. Two versions are currently being implemented. The first version supports an unlimited number of symbolic variables but generates answers that are symbolic only in the poles (the powers of the exponential terms). The coefficients of the polynomials in t that multiply the exponentials are numeric. The second version of ACE is fully symbolic in one variable and numeric in other variables, i.e. the coefficients of the polynomials in t that multiply the exponentials have both numeric and symbolic parts. This allows us to conduct a parametric sensitivity analysis in a fully symbolic fashion. Eventually these two methods will be combined yielding completely symbolic poles and coefficients that are fully symbolic in at least one variable.

Several problems have arisen in constructing the ACE package. When computing symbolic coefficients, the size of the coefficients grows linearly with the number of symbolic variables used along all paths to the state. The coefficients rapidly reach an unmanageable size, even for a small chain. Restricting the lengths of the paths through the chain would greatly reduce the package's utility, particularly for chains that are "long" (e.g. simple death processes). Instead, we restrict the number of symbolic variables that are maintained in a given run of the program. All variables not treated symbolically are merged numerically. If poles are still maintained in a fully symbolic fashion, care must be taken to correctly merge the numeric values of symbolically different, numerically identical poles.

Further efforts, include the construction of a "user-friendly" interface and the addition of a block definition and solution facility. The user will be able to define blocks of states with fixed entry and exit points. The blocks could be evaluated by direct insertion of their states into the chain. Alternatively, the block could be solved in isolation using symbolic or numerical approximation methods. This capability should further facilitate the use of ACE in evaluating aggregation methods.

4. Examples and Conclusions

In this section we demonstrate the use of ACE-like symbolic computation. We begin by symbolically solving the example given in Figure 2 using the method described in sections 2 and 3. We then apply a simple aggregation technique to the chain and resolve the system. We give examples that demonstrate the utility of a symbolic solution for bounding, sensitivity analysis, and comparison of aggregation techniques.

4.1 Exact Solution of 3-Component System

Given the chain shown in Figure 2, we follow the algorithm outline given in section 3. We first observe that the only γ for the probability distribution of state 3 is -3λ . As

state 3 has no parent the constant $\alpha'_{1,0}=1$. Accordingly we can write:

$$P_3(t) = e^{-\lambda t} \quad (20)$$

We can continue following the algorithm sketch and derive the following equations for the functioning states:

$$P_{2R}(t) = -\frac{3\lambda}{\lambda-\delta} e^{-\lambda t} + \frac{3\lambda}{\lambda-\delta} e^{-(2\lambda+\delta)t} \quad (21)$$

$$P_2(t) = \frac{3\delta}{\lambda-\delta} e^{-\lambda t} - \frac{3\lambda}{\lambda-\delta} e^{-(2\lambda+\delta)t} + 3e^{-2\lambda t} \quad (22)$$

$$P_{1R}(t) = -\frac{6\lambda\delta}{(\lambda-\delta)(2\lambda-\delta)} e^{-\lambda t} + \frac{6\lambda}{\lambda-\delta} e^{-(2\lambda+\delta)t} - \frac{6\lambda}{\lambda-\delta} e^{-2\lambda t} + \frac{6\lambda\delta}{(\lambda-\delta)(2\lambda-\delta)} e^{-(\lambda+\delta)t} \quad (23)$$

$$P_1(t) = \frac{3\delta^2}{(\lambda-\delta)(2\lambda-\delta)} e^{-\lambda t} - \frac{6\lambda\delta}{(\lambda-\delta)(\lambda+\delta)} e^{-(2\lambda+\delta)t} + \frac{6\delta}{\lambda-\delta} e^{-2\lambda t} - \frac{6\lambda\delta}{(\lambda-\delta)(2\lambda-\delta)} e^{-(\lambda+\delta)t} + \frac{3\delta}{(\lambda+\delta)} e^{-\lambda t} \quad (24)$$

For the state that corresponds to failure due to exhaustion of components we can write:

$$P_{F_e}(t) = -\frac{\delta^2}{(\lambda-\delta)(2\lambda-\delta)} e^{-\lambda t} + \frac{6\lambda^2\delta}{(\lambda-\delta)(\lambda+\delta)(2\lambda+\delta)} e^{-(2\lambda+\delta)t} - \frac{3\delta}{\lambda-\delta} e^{-2\lambda t} + \frac{6\lambda^2\delta}{(\lambda-\delta)(\lambda+\delta)(2\lambda-\delta)} e^{-(\lambda+\delta)t} - \frac{3\delta}{\lambda+\delta} e^{-\lambda t} + \frac{\delta^2}{(2\lambda+\delta)(\lambda+\delta)} \quad (25)$$

For the state that corresponds to a coverage failure we can write:

$$P_{F_c}(t) = \left(\frac{2\lambda}{\lambda-\delta} + \frac{2\lambda\delta}{(\lambda-\delta)(2\lambda-\delta)}\right) e^{-\lambda t} - \frac{12\lambda^2}{(\lambda-\delta)(2\lambda+\delta)} e^{-(2\lambda+\delta)t} + \frac{3\lambda}{\lambda-\delta} e^{-2\lambda t} - \frac{6\lambda^2\delta}{(\lambda-\delta)(\lambda+\delta)(2\lambda-\delta)} e^{-(\lambda+\delta)t} + \frac{2\lambda}{2\lambda+\delta} + \frac{\lambda\delta}{(2\lambda+\delta)(\lambda+\delta)} \quad (26)$$

We note that the reliability of the 3-component system is given by

$$R(t) = 1 - (P_{F_e}(t) + P_{F_c}(t)) \quad (27)$$

To derive the information contained in this symbolic reliability expression, even a highly flexible conventional reliability evaluation package would require several runs for different parameter values. For example, to see the effect of the fault-handling rate we consider the reliability expression as a function of delta. Fixing $\lambda=10^{-4}$

failure/hour and $t=10$ hours, in Figure 3 we graph $-\log_{10}$ of the system unreliability as a function of $-\log_{10}$ of the mean time to handle a fault. When faults are handled quickly (in this case about 100 ms), we see that the reliability of the system approaches that of a system with perfect coverage. When fault-handling is slow (minutes or hours), imperfect coverage dramatically reduces the system's reliability.

4.2 An Approximate Solution of the 3-Component System

When Markov models are used for realistic systems, the state space often grows beyond practical limits. Accordingly, reliability evaluation packages often use various aggregation or lumping methods to reduce the size of the state space. For example, a system model can be decomposed into sub-models of fault-handling and fault-occurrence behavior.¹⁶ Short of assuming all faults are successfully handled, one of the simplest approaches is to condense the second fault rate and fault-handling parameters into a single constant c denoting *coverage*, the probability that an arbitrary fault is successfully handled. When this approach is used with our example, we obtain the chain shown in Figure 4. Its state probability equations are

$$P_{3A}(t) = e^{-3\lambda t} \quad (28)$$

$$P_{2A}(t) = -3c_1 e^{-3\lambda t} + 3c_1 e^{-2\lambda t} \quad (29)$$

$$P_{1A}(t) = 3c_1 c_2 e^{-3\lambda t} - 6c_1 c_2 e^{-2\lambda t} + 3c_1 c_2 e^{-\lambda t} \quad (30)$$

$$P_{F_0}(t) = -c_1 c_2 e^{-3\lambda t} + 3c_1 c_2 e^{-2\lambda t} - 3c_1 c_2 e^{-\lambda t} + c_1 c_2 \quad (31)$$

$$P_{F_1}(t) = (2(1-c_2)c_1 - (1-c_1))e^{-3\lambda t} - 3c_1(1-c_2)e^{-2\lambda t} + (1-c_1) + c_1(1-c_2) \quad (32)$$

As in the original chain,

$$R(t) = 1 - (P_{F_0}(t) + P_{F_1}(t))$$

One interesting problem is correctly choosing values for the c parameter. If two c parameters were employed, as in our example, the instantaneous coverage approximation would usually be

$$c_1 = \frac{\delta}{2\lambda + \delta} \quad (33)$$

$$c_2 = \frac{\delta}{\lambda + \delta} \quad (34)$$

However, if only a single coverage parameter were chosen, the choice might well depend on the period of time over which we were interested in evaluating the reliability of the system.

With the reliability expressions for both the aggregated and original Markov chains, we can evaluate the acceptability of the aggregation scheme by comparing the results they produce. Figure 4 shows the graphs of three estimates of $-\log_{10}$ of system unreliability as a function of \log_{10} of t . With λ fixed at 10^{-4} failures/hr. for all three curves, the lower curve is derived by solving the original Markov model in Figure 2 with $\delta = 1$. Using this δ value, the middle curve is derived using the aggregated chain in Figure 4 and instantaneous coverage estimates derived from equations (33) and (34). The upper curve is derived using a naive *perfect coverage* model, i.e. fault handling is assumed to always succeed instantaneously. Even for this contrived situation (λ and δ are probably much closer in value than they would be in practice), we see that a constant coverage assumption still can provide a good estimate of system reliability. For our particular example, if a more realistic δ value is chosen, the reliability estimates provided by the original and aggregated chains are essentially identical. Extending this validation of a simple approximation scheme for a small model to more realistic models may require significant effort.

Symbolic solutions of CTMC should provide at least two benefits. First, it should be possible to compare the results obtained by exact and approximate solution methods for small to medium sized CTMC. By indicating the magnitude of error that approximate solutions introduce, this type of analysis should provide a good indication of an aggregation/approximation technique's utility for larger, more realistic problems. Second, symbolic solutions allow us to easily examine the influence of changing parameter values on the solutions of Markov models. This type of

investigation could be very expensive using conventional simulation or numerical solution techniques. By providing easy access to symbolic solutions of CTMC, the ACE package should enhance our ability to study Markov reliability models. and approximation techniques for their solution.

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Appendix - The ACE Procedure

GIVEN: An acyclic Markov chain with state space $\{1, 2, \dots, N\}$ sorted so that the ancestors of a every state precede it in the list (for convenience).

BEGIN

FOR each state $s \in \{1, 2, \dots, N\}$

Determine $J(s)$ the set of states with transitions leading to s

Determine $S(J(s))$ and $\{L_i(\gamma) : \gamma \in S(J(s))\}$ using (9) and (15)

Compute $a_{\gamma,k}$ $k=0, \dots, L_i(\gamma)$, $\gamma \in S(J(i))$

Let $\gamma^* = g_i$ and $a_{\gamma^*,0} = 0$

Note: γ^* for an absorbing state is 0.

FOR $\gamma \in S(J(i))$, $\gamma \neq \gamma^*$

FOR $k=0, 1, \dots, L_i(\gamma)$

Compute $a_{\gamma,k}$ using formula (17)

Accumulate $a_{\gamma^*,0} = a_{\gamma^*,0} + (-1)^{k+1} \cdot k! \frac{a_{\gamma,k}}{(\gamma - \gamma^*)^{k+1}}$

END FOR

END FOR

If $\gamma^* \in S(J(i))$, compute $a_{\gamma^*,k+1}$ from (19) for $k=0, \dots, L_i(\gamma^*)$

$$P_s(t) = \sum_{\gamma \in S(J(i))} e^{\gamma t} \sum_{k=0}^{L_i(\gamma)} a_{\gamma,k} t^k$$

END FOR

END

Method: Simulation

Package: CAST¹⁷

Note: Allows full simulation of a restricted class of systems

Method: Differential Equations (Numerical Solution)

Package: HARP⁵

Domain: Homogeneous and Non-Homogeneous acyclic CTMC

Package: SAVE^{*}

Domain: Cyclic CTMC

Method: Differential Equations (Laplace Solution)

Package: SURF¹¹

Domain: Non-Markov Processes

Note: Approximate solution using Coxian method of stages

Method: Integral Equations (Numerical Solution)

Package: Care III Coverage Model¹³

Domain: Semi-Markov Processes

Package: Care III Reliability Model¹³

Domain: Non-Homogeneous CTMC

Method: Closed Form Solution

Package: ARIES¹⁸

Domain: Cyclic Homogeneous CTMC

Note: Poles and their coefficients derived numerically

Package: ACE

Domain: Acyclic Homogeneous CTMC

Note: Poles and their coefficients derived symbolically

Figure 1: Reliability Modeling Packages Employing Markov Chain Techniques

^{*}System Availability Estimator; in preparation under an IBM-Duke Joint Study.

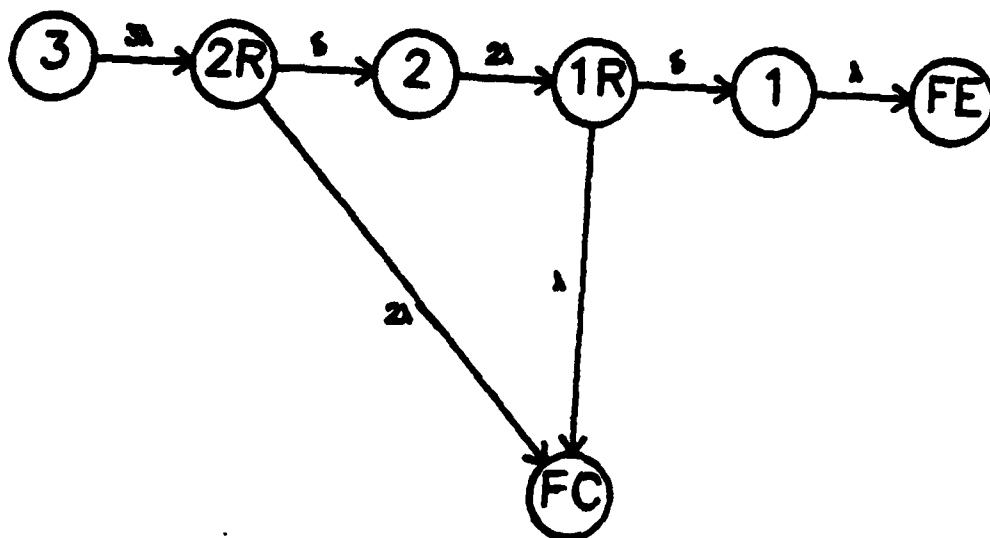


Figure 2: Markov Chain State Diagram for a 3-Component Parallel Redundant System
Coverage failure results only from a second fault occurrence during reconfiguration

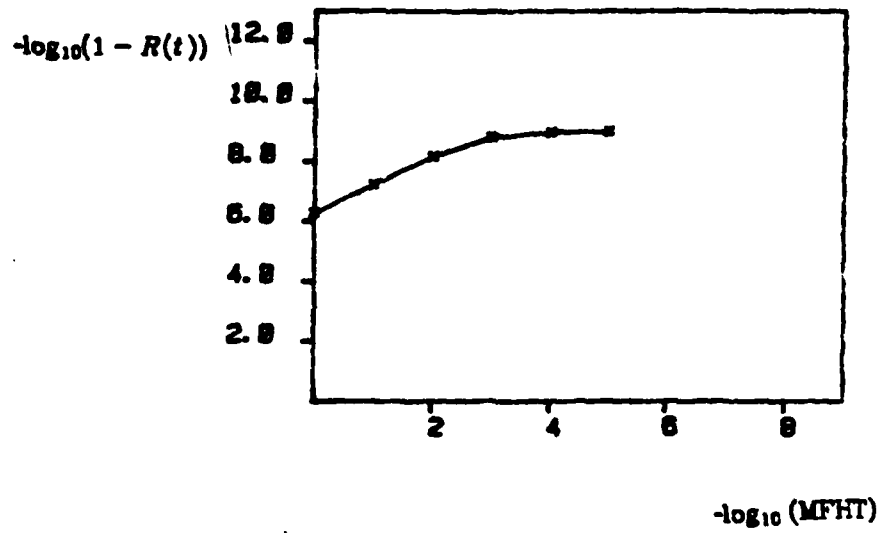


Figure 3: Effect of Fault-Handling Rate on Reliability

Mission Time = 10 hrs. $\lambda = 10^{-4}$

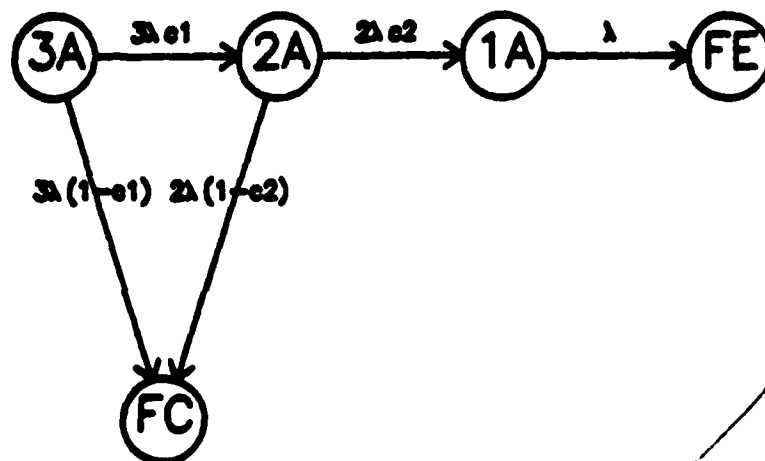


Figure 4: Aggregated Markov Chain State Diagram for 3-Component Parallel Redundant System
 C_1 and C_2 are instantaneous coverage estimates

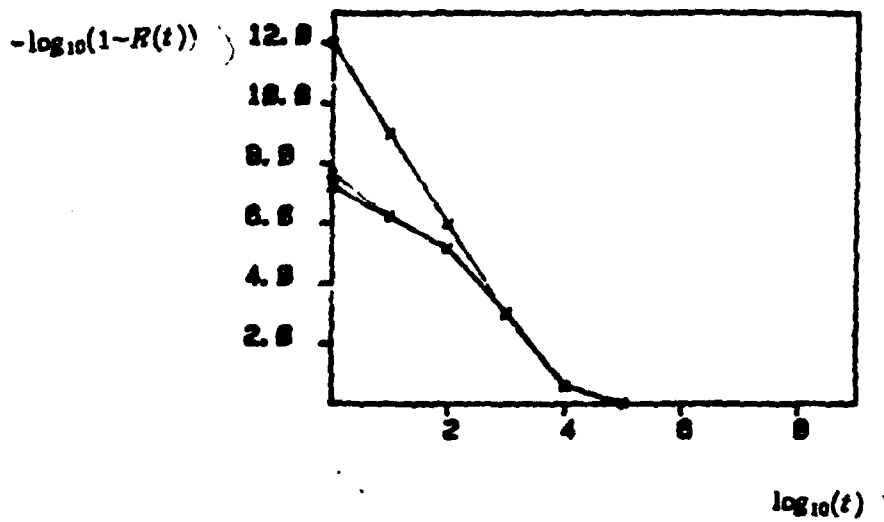
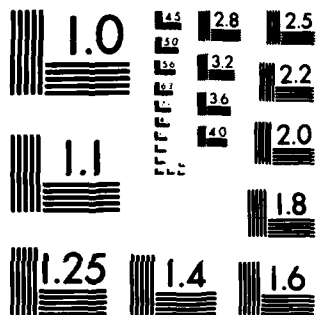


Figure 5: Aggregation's Effect on Reliability Estimates

The bottom curve is an estimate of reliability derived from the original Markov chain.

The middle curve is derived from the aggregated chain in Figure 4

The top curve is a *perfect* coverage estimate



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